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Analytical Approximate Solutions of Stochastic Models Arising in Statistical Systems and Financial Markets

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Abstract: Homotopy Perturbation Method (HPM) is applied to solve stochastic models and in the simialr context Fokker-Planck equation for non-equilibrium statistical systems and Black-Scholes model for pricing stock options are tackled The analytical solutions are calculated in the form of convergent power series. The results reveal that HPM is very effective and convenient for stochastic models.

Key words: Homotopy perturbation method . statistical systems . stohastic analysis . fokker planck equation . black-scholes model

INTRODUCTION

Stochastic partial differential equations are essentially partial differential equations that have additional random terms. [1] which are usually very difficult to solve, either numerically or theoretically. However, they have strong connections with quantum field theory and statistical mechanics [1]. The study of stochastic partial differential equations bring together techniques from probability theory, functional analysis and the theory of partial differential equations. Stochastic partial differential equations have wide applications in various fields of sciences: study of random evolution of systems with a spatial extension (random interface growth, random evolution of surfaces, fluids subject to random forcing), study of stochastic models where the statevariable is infinite dimensional (for example, a curve or surface). The solutions to stochastic partial differential equations may be viewed in several manners [2, 3]. One can view a solution as a random field (set of random variables indexed by a multidimensional parameter). An imortant stochastic model is the Fokker-Planck equation whihc arises in various fields in natural science, including solid-statephysics, quantum optics, chemical physics, theoretical biology and circuit theory. The Fokker-Planck equation was first used by Fokker and Plank [4] to describe the Brownian motion of particles. If a small particle of mass m is immersed in a fluid, the equation of motion for the distribution function F(x, t) is given by

$$\frac{\partial F}{\partial t} = \gamma\frac{\partial F}{\partial v} + \gamma\frac{KT}{m}\frac{\partial^2 F}{\partial v^2} \tag{1.1}$$

where v is the velocity for the Brownian motion of a small particle, t is the time, γ is the fraction constant, K is the Boltzmann's constant and T is the temperature of the fluid [4]. Eq. (1.1) used in the force filed for studying stabilities of a collisions plasma and Brownian particle moving through a medium at some fixed temperature.

A more general form of FPE, is the nonlinear FPE, which has important applications in various areas, such as, plasma physics, population dynamics, biophysics, engineering, laser physics and marketing [5]. In one variable case, the nonlinear Fokker-Planck equation is written in the following form [4].

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$$\frac{\partial u}{\partial t}=\left[-\frac{\partial A(x,t,u)u}{\partial x}+\frac{\partial^2 B(x,t,u)u}{\partial x^2}\right] \tag{1.2}$$

with the initial condition given by

$$u(x,0)=f(x), x\in R \tag{1.3}$$

where u(x,t) is the unknown distribution function. In Eq. (1.2), B(x,t,u)>0 is called the diffusion coefficient and A(x,t,u)>0 is called the drift coefficient. A special case of Eq. (1.2) is the FPE from plasma physics which has the form [6]

$$\frac{\partial p}{\partial t}=\frac{1}{2}\frac{\partial^2}{\partial x^2}\left(\frac{1}{2}x^{1-2\varepsilon}p\right)-\frac{\partial}{\partial x}\left(\frac{1}{4}x^{-2\varepsilon}p\right) \tag{1.4}$$

p(x, t) is the probability density function. Making use of the replacement

$$p(x,t)=\sqrt{2}x^{(2\varepsilon-1)/2}\omega(\tau y), \quad \tau=t, y=\frac{2\sqrt{2}}{2\varepsilon+1}x^{(2\varepsilon+1)/2} \tag{1.5}$$

Eq. (1.4) reduces to the FPE for the linear Brownian motion as

$$\frac{\partial\omega(\tau y)}{\partial\tau}=-\frac{\partial}{\partial y}\left(\frac{\omega}{2y}\right)\frac{1}{2}\frac{\partial^2\omega}{\partial y^2} \tag{1.6}$$

with initial condition

$$\omega(0,y)=\text{erf}\left[\frac{1}{2}\sqrt{2}y\right]e^{y^2/2}+e^{y^2/2} \tag{1.7}$$

The stochastic analysis has interesting applications in mathematical modeling of financial market option pricing [6]. In option pricing theory, the Black-Scholes equation for the determination of the fair value of a call option or derivative security on the market, is one of the most effective models [7, 8]. For European options, the Black-Scholes equation can be solved in terms of a diffusion equation boundary value problem, or directly using the Mellin transform [1].

The most well-known stochastic model for the equilibrium condition between the expected return on the option, the expected return on the stock and the riskless interest rate is the Black-Scholes equation [9]

$$\frac{\partial C(s,t)}{\partial t}=\frac{v^2}{2}s^2\frac{\partial^2 C(s,t)}{\partial s^2}+rs\frac{\partial C(s,t)}{\partial s}-rC(s,t) \tag{1.8}$$

where s is the assert price, which undergoes geometric Brownian motion C(s,t) is the call price, v is the velocity and r is the reckless interest rate.

It is to be highlighted that several tecniques including HPM, Adomian’s Decomposition (ADM), Variational Iteration, Semidiscrete Glariken have been used to solve some stochastic models [2, 11-16] and the refernces therein. The basic motivation of this paper is the extension of a relativel new technique which is called Homotopy Perturbation Method (HPM) for analytic and approximate solutions of some stochastic models. It is ibserved that the proposed scheme is fully compatible with the physical nature of such problems and the same may be exteneded to other physical models also.

DESCRIPTION OF METHOD

Consider the following nonlinear differential equation [12-14]:

$$A(u)-f(r)=0, \; r\in\Omega \tag{1.9}$$

with boundary conditions

$$B\left(u,\frac{\partial u}{\partial n}\right)=0, r\in\Gamma \quad (1.10)$$

where A is a general differential operator, B is a boundary operator, f (r) is a known analytic function, G is the boundary of the domain O.

The operator A can, generally speaking, be divided into two parts L and N, where L is linear and N is nonlinear; therefore, Equation (1.9) can be written as,

$$L(u)+N(u)-f(r)=0 \quad (1.11)$$

By using homotopy technique, one can construct a homotopy $v(r,p): \Omega \times [0,1] \to \mathcal{R}$ which satisfies

$$H(v,p)=(1-p)[L(v)-L(u_0)]+p[A(v)-f(r)]=0 \quad p\in[0,1] \quad (1.12a)$$

or

$$H(v,p)=L(v)-L(u_0)+pL(u_0)+p[N(v)-f(r)]=0 \quad (1.12b)$$

where $p\in[0,1]$ is an embedding parameter and u_0 in the initial approximation of Eq. (1.9) which satisfies the boundary conditions. Clearly, we have

$$H(v,0)=L(v)-L(u_0)=0 \quad (1.13)$$

$$H(v,1)=A(v)-f(r)=0 \quad (1.14)$$

The changing process of p from zero to unity is just that of v(r,p) changing from $u_0(r)$ to u(r). This is called deformation and also L(v)-L(u_0) and A(v)-*f*(v) are called homotopic intopology. If, the embedding parameter p;($0\le p\le 1$) is considered as a "small parameter,"applying the classical perturbation technique, we can naturally assume that the solution of Eqs.(1.13) and (1.14) can be given as a power series in p, i.e.,

$$v=v_0+pv_1+p^2v_2+\cdots$$

and setting p = 1 results in the approximate solution of Eq. (1.12) as;

$$u=\lim_{p\to 1} v=v_0+v_1+v_2+\cdots \quad (1.16)$$

APPLICATIONS

The Fokker-Planck Equaiton

Example 3.1: [14]

In Eq. (1.2):

$$A(x,t,u)=\frac{2-\frac{x}{2}}{t+x} \quad (1.17)$$

and

$$B(x,t,u)=1 \quad (1.18)$$

Consider (1.3) with :

$$u(x,0)=f(x)=x, x\in R \quad (1.19)$$

We rewrite Eq.(1.2) as

$$\frac{\partial u}{\partial t}=\left[-\frac{\partial\left[\dfrac{2-\dfrac{x}{2}}{t+x}u\right]}{\partial x}+\frac{\partial^2 u}{\partial x^2}\right] \tag{1.20}$$

With the initial condition

$$u_0(x,0)=u(x,0)=x \tag{1.21}$$

To solve Equations (1.20)-(1.21) by HPM

$$\frac{\partial u}{\partial t}-\frac{\partial u_0}{\partial t}=p\left[-\frac{\partial\left[\dfrac{2-\dfrac{x}{2}}{t+x}u\right]}{\partial x}+\frac{\partial^2 u}{\partial x^2}-\frac{\partial u_0}{\partial t}\right] \tag{1.22}$$

Assume the solution of Equation (1.22) in the form:

$$u=u_0+pu_1+p^2u_2+p^3u_3+\ldots \tag{1.23}$$

Substituting (1.23) into Equation (1.22) and collecting terms of the same power of p gives:

$$p^0:\frac{\partial u_0}{\partial t}-\frac{\partial u_0}{\partial t}=0$$

$$p^1:\frac{\partial u_1}{\partial t}=\left(\frac{1}{2}\frac{x\left(\dfrac{\partial u_0}{\partial x}\right)}{(t+x)}+\frac{2u_0}{(t+x)^2}-\frac{2\left(\dfrac{\partial u_0}{\partial x}\right)}{(t+x)}+\frac{1}{2}\frac{u_0}{(t+x)}-\left(\frac{\partial u_0}{\partial t}\right)-\frac{1}{2}\frac{xu_0}{(t+x)^2}+\frac{\partial^2 u_0}{\partial x^2}\right)$$

$$p^2:\frac{\partial u_2}{\partial t}=\left(-\frac{2\left(\dfrac{\partial u_1}{\partial x}\right)}{(t+x)}-\frac{1}{2}\frac{xu_1}{(t+x)^2}+\frac{1}{2}\frac{u_1}{(t+x)}+\left(\frac{\partial^2 u_1}{\partial x^2}\right)+\frac{2u_1}{(t+x)^2}+\frac{1}{2}\frac{x\left(\dfrac{\partial u_1}{\partial x}\right)}{(t+x)}\right)$$

$$p^3:\frac{\partial u_3}{\partial t}=\left(-\frac{2\left(\dfrac{\partial u_2}{\partial x}\right)}{(t+x)}+\frac{1}{2}\frac{u_2}{(t+x)}+\frac{1}{2}\frac{x\left(\dfrac{\partial u_2}{\partial x}\right)}{(t+x)}-\frac{1}{2}\frac{xu_2}{(t+x)^2}+\frac{2u_2}{(t+x)^2}+\left(\frac{\partial^2 u_2}{\partial x^2}\right)\right)$$

$$\vdots$$

The solution reads

$$u_1(x,t)=x\ln(t+x)-\frac{2x}{t+x}-2\ln(t+x)+\frac{1}{2}\frac{x^2}{t+x}-x \tag{1.24}$$

$$\begin{aligned}u_2(x,t)=&-\frac{10}{(t+x)}-\frac{5}{4}\frac{x^2}{(t+x)}-\frac{5}{2}\frac{x^2}{(t+x)^2}+\frac{6x}{(t+x)^2}+\frac{1}{2}\frac{x^2\ln(t+x)}{(t+x)}\\&-\frac{3x\ln(t+x)}{(t+x)}+\frac{9x}{(t+x)}+2\ln(t+x)+\frac{1}{4}\frac{x^3}{(t+x)^2}+\frac{4\ln(t+x)}{(t+x)}\end{aligned} \tag{1.25}$$

$$u_3(x,t) = -\frac{3x^2\ln(t+x)}{(t+x)^2} + \frac{52}{(t+x)^2} + \frac{43}{(t+x)} + \frac{9}{8}\frac{x^2}{(t+x)} - \frac{53}{4}\frac{x}{(t+x)^3} - \frac{27\ln(t+x)}{(t+x)}$$
$$-\frac{12\ln(t+x)}{(t+x)^2} + \frac{25}{2}\frac{x\ln(t+x)}{(t+x)} + \frac{11x\ln(t+x)}{(t+x)^2} - \frac{5}{4}\frac{x^2\ln(t+x)}{(t+x)} + 4\ln(t+x) - \frac{7}{8}\frac{x^3}{(t+x)^2} \tag{1.26}$$
$$+\frac{1}{4}\frac{x^3\ln(t+x)}{(t+x)^2} - \frac{9}{2}\ln(2t+2x) + \frac{1}{8}\frac{x^4}{(t+x)^3} - \frac{33}{2}\frac{x}{(t+x)} - \frac{53x}{(t+x)^2} + \frac{103}{8}\frac{x^2}{(t+x)^2}$$
$$\vdots$$

and so on.
Hence,we have

$$u(x,t) = \frac{1}{(t+x)^3}\left(-\frac{53}{4}x + \frac{1}{8}x^4\right) + \frac{1}{(t+x)^2}\left(\ln(t+x)\left[\frac{1}{4}x^3 - 3x^2 + 11x - 12\right] - \frac{5}{8}x^3 + \frac{83}{8}x^2 - 47x + 52\right)$$
$$+\frac{1}{(t+x)}\left(\ln(t+x)\left[-\frac{3}{4}x^2 + \frac{19}{2}x - 23\right] + \frac{3}{8}x^2 - \frac{19}{2}x + 33\right) + \ln(t+x)\left[-x - \frac{1}{2}\right] - \frac{9}{2}\ln 2 \tag{1.27}$$

which is the approximate solution of equation.This approximate solution is shown in Fig. 1.

Example 3.2: Consider the following Fokker-Planck equation which arises [6]
We find the solution of Eq.(1.4) by HPM. We simply solve Eq.(1.6).

$$\frac{\partial w}{\partial \tau} - \frac{\partial w_0}{\partial \tau} = p\left(-\frac{1}{2}\frac{\frac{\partial w}{\partial y}}{y} + \frac{1}{2}\frac{w}{y^2} + \frac{1}{2}\frac{\partial^2 w}{\partial y^2} - \frac{\partial w_0}{\partial \tau}\right) \tag{1.28}$$

$$w_0 = \operatorname{erf}\left(\frac{1}{2}\sqrt{2}y\right)e^{\frac{1}{2}y^2} + e^{\frac{1}{2}y^2} \tag{1.29}$$

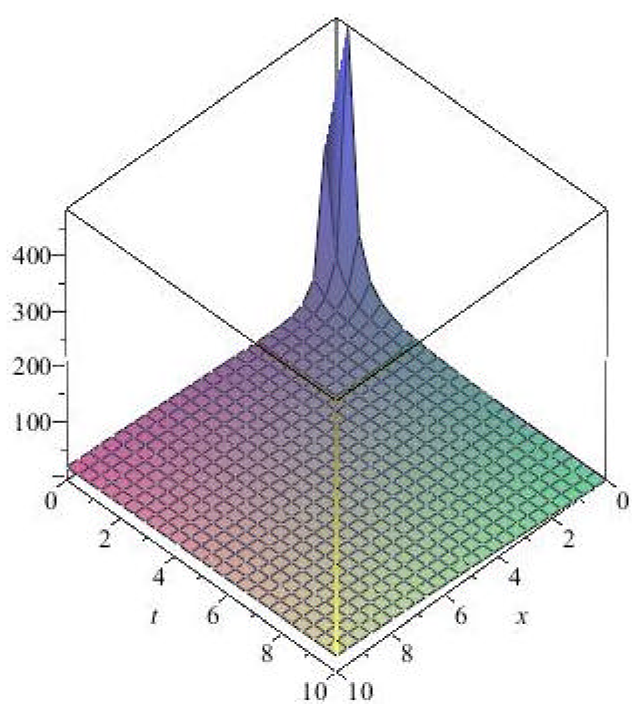

Fig. 1: Approximate series solution of example 3.1

$$w_1 = e^{\frac{y^2}{2}}\frac{\tau}{2}\left(-\left(y-\frac{1}{y}\right)\frac{\sqrt{2}e^{\frac{-y^2}{2}}}{\sqrt{\pi}}+\left(\frac{1}{y^2}+y^2\right)\left(\operatorname{erf}\left(\frac{1}{2}\sqrt{2}y\right)+1\right)\right) \tag{1.30}$$

$$w_2 = \frac{e^{\frac{y^2}{2}}\tau^2}{8}\left((3y-7)e^{\frac{-y^2}{2}}\frac{\sqrt{2}}{\sqrt{\pi}}+\left(\frac{7}{y^4}-\frac{1}{y^2}+3y^2+3\right)\left(\operatorname{erf}(\frac{1}{2}\sqrt{2}y)+1\right)\right) \tag{1.31}$$

$$\vdots$$

$$\begin{aligned} w(\tau,y) = {} & \frac{e^{\frac{y^2}{2}}\tau^2}{8}\left((3y-7)e^{\frac{-y^2}{2}}\frac{\sqrt{2}}{\sqrt{\pi}}+\left(\frac{7}{y^4}-\frac{1}{y^2}+3y^2+3\right)\left(\operatorname{erf}(\frac{1}{2}\sqrt{2}y)+1\right)\right) \\ & + e^{\frac{y^2}{2}}\frac{\tau}{2}\left(-\left(y-\frac{1}{y}\right)\frac{\sqrt{2}e^{\frac{-y^2}{2}}}{\sqrt{\pi}}+\left(\frac{1}{y^2}+y^2\right)\left(\operatorname{erf}\left(\frac{1}{2}\sqrt{2}y\right)+1\right)\right)+\operatorname{erf}\left(\frac{1}{2}\sqrt{2}y\right)e^{\frac{1}{2}y^2}+e^{\frac{1}{2}y^2} \end{aligned} \tag{1.32}$$

The behavior of the probability density p(x, t) versus x for different values of time is shown in Fig. 2.

Example 3.3: Itõ stochastic differential equation.

The Fokker-Planck equation can be used for computing the probability densities of other stochastic differential equations. Consider Itõ stochastic differential equation

$$dX_t = \mu(X_t,t)\,dt + \sigma(X_t,t)\,dW_t \tag{1.33}$$

where $X_t \in R^N$ is the state and $W_t \in R^M$ is a standard M-dimensional Wiener process. If the initial distribution is $X_0 \in p(x,0)$ then the probability density p(x,t) of the state X_t is given by the FP equation (3.4) with $F_i(x,t) = \mu_i(x,t)$ and diffusion terms

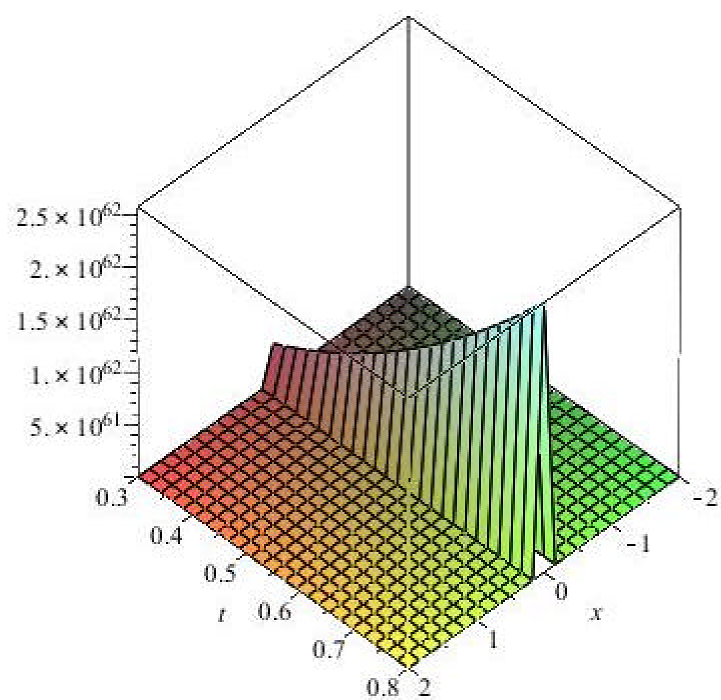

Fig. 2: The behavior of the probability density function p(x, t) in Example 3.2

$$D_{ij} = \frac{1}{2}\sum_k \sigma_{ik}(x,t)\sigma_{kj}^T(x,t)$$

A standard scalar Wiener process is generated by the stochastic differential equation

$$dX_t = dW_t$$

Now the drift term μ is zero and the diffusion coefficient is 1/2 and thus the corresponding FP equation is the simplest form of diffusion equation

$$\frac{\partial p(x,t)}{\partial t} = \frac{\partial^2 p(x,t)}{\partial x^2}$$

The black-scholes equation: Stochastic analysis have interesting applications in mathematical modelling and financial market option pricing. The most well-known stochastic model for the equilibrium condition between the expected return on the option, the expected return on the stock and the riskless interest rate is the Black-Scholes equation (1.8). Following [6], we reformulate (1.8) by introducing a new dependent variable

$$x = \ln s, p(x,t) = e^{rt}C(s,t)$$

where p(x,t) is the probability density function. As a result, Eq. (1.8) transfers to a diffusion convection-reaction equation of Brownian motion.

Example 3.4: The Black-Scholes model [9].

The Black-Scholes model for time evolution of the call price option C(s, t), as a function of the underlying asset price s and time t, is given by Eq. (1.8). To solve (1.8) by HPM, the correction functional reads as

$$\frac{\partial C}{\partial t} - \frac{\partial C_0}{\partial t} = p\left(\frac{1}{2}v^2 s^2\left(\frac{\partial^2 C}{\partial s^2}\right) + rs\frac{\partial C}{\partial s} - rC\right) \tag{1.34}$$

$$C_0 = s + \frac{1}{s^{7/5}} \tag{1.35}$$

$$C_1 = \left(\frac{42}{25}\frac{v^2}{s^{7/5}} - \frac{12}{5}\frac{r}{s^{7/5}}\right)t \tag{1.36}$$

$$C_2 = \left(\frac{1764}{625}\frac{v^4 t}{s^{7/5}} - \frac{1008}{125}\frac{v^2 tr}{s^{7/5}} + \frac{144}{25}\frac{r^2 t}{s^{7/5}}\right)t \tag{1.37}$$

$$C_3 = \left(\frac{74088}{15625}\frac{v^6 t^2}{s^{7/5}} - \frac{63504}{3125}\frac{v^4 t^2 r}{s^{7/5}} + \frac{18144}{625}\frac{v^2 t^2 r^2}{s^{7/5}} - \frac{1728}{125}\frac{t^2 r^3}{s^{7/5}}\right)t \tag{1.38}$$

$$\vdots$$

$$\begin{aligned} C(s,t) = 3s + \frac{2}{s^{7/5}} + \frac{84}{25}\frac{v^2 t}{s^{7/5}} - \frac{12}{5}\frac{tr}{s^{7/5}} + \frac{147}{52}\frac{v^4 t^2}{s^{7/5}} - \frac{1008}{125}\frac{v^2 t^2 r}{s^{7/5}} + \frac{144}{25}\frac{t^2 r^2}{s^{7/5}} \\ - \frac{17}{5}rt + s^{7/5} + \frac{74088}{15625}\frac{v^6 t^3}{s^{7/5}} - \frac{63504}{3125}\frac{v^4 t^3 r}{s^{7/5}} + \frac{18144}{625}\frac{v^2 t^3 r^2}{s^{7/5}} - \frac{1728}{125}\frac{t^3 r^3}{s^{7/5}} \end{aligned} \tag{1.39}$$

p(x,t) is computed for $v = 0.2, r = 0.01, \varepsilon = \frac{1}{2}$ and shown Fig. 3.

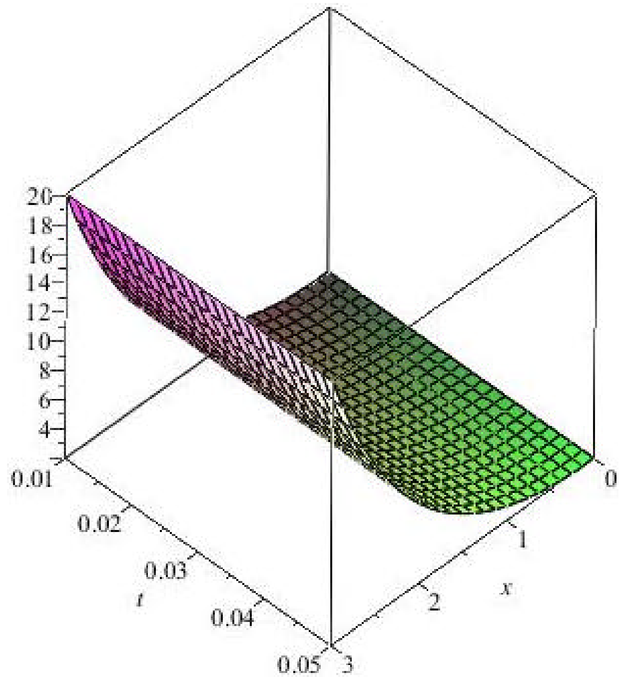

Fig. 3: The behavior of the probability density function p(x, t) in Example 3.4

CONCLUSIONS

Homotopy Perturbation Method (HPM) has been successfully applied to linear and nonlinear stochastic models. Numerical results are fully supporitve of the efficiency and reliability of proposed algorithm.

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